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# $M$-theory on the Stiefel manifold and 3d conformal field theories

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Abstract: We compute the mass and multiplet spectrum of $M$–theory compactified on the product of $AdS_4$ spacetime by the Stiefel manifold $V_{(5,2)} = \mathrm{SO}(5)/\,\mathrm{SO}(3)$, and we use this information to deduce via the AdS/CFT map the primary operator content of the boundary $\mathcal{N} = 2$ conformal field theory. We make an attempt for a candidate supersymmetric gauge theory that, at strong coupling, should be related to parallel $M2$-branes on the singular point of the non-compact Calabi-Yau four-fold $\sum_{a=1}^{5} z_a^2 = 0$, describing the cone on $V_{(5,2)}$.

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## Contents



## 1. Introduction

Duality between large-$N$ field theories on D3-branes placed at conical singularities and type-IIB theory compactifications on a 5d manifold has been the subject of many investigations in the last couple of years [1]–[3] (for a rewiew see [4]). In particular, in [5] an $\mathcal{N} = 1$ superconformal Yang-Mills field theory has been constructed which turned out to be dual to type-IIB theory compactified on the $T^{11} = \mathrm{SU}(2)^2/\mathrm{U}(1)$ manifold. Solid evidence on such duality was developed in [6, 7, 8] and in [9], where a one-to-one correspondence was established between the Kaluza-Klein (KK) supergravity spectrum on $T^{11}$ [10] and the boundary superconformal operators. In the case of $M2$ and $M5$ branes the AdS/CFT conjecture was first studied for maximal supersymmetry in compactifications on spheres $S^7$ and $S^4$ [11, 12, 13] and then further extended to lower $\mathcal{N}$ for sphere-orbifolds and other brane systems [14, 15].

In [16] the question was raised whether one could similarly construct a SuperConformal Field Theory (SCFT) in three dimensions which is dual for large brane-flux $N$ to $M$-theory compactified on $AdS_4 \times M_7$ [16, 17, 18], where $M_7$ is one among the compact Einstein manifolds that were classified [19] and analysed in the eighties [20, 21]. Such three-dimensional SCFT's would then be defined as the conformal limit of the world-volume theory of N M2-branes positioned at a conical singularity of $M_3 \times Y_8$, where $Y_8$ is the cone over $M_7$.

Three-dimensional SCFT's are more difficult to analyze because they emerge in non-perturbative limit of conventional gauge quantum field theories. The origin of this difficulty is well known, and can be traced back to the fact that the three dimensional bare gauge coupling constant is dimensionful, so that a conformal description of the theory dual to supergravity is only possible in the infrared limit, where the gauge coupling blows up and the gauge fields are integrated out.

In this regime, the SCFT is described as the low energy theory of N coincident M2-branes, where N is the number of units of flux of the dual of the eleven-dimensional four-form on the internal manifold $M_7$. Indeed, compactifications of $M$-theory on a circle leads to a D2-brane description in type-IIA theory whose world-volume gauge theory is not conformal and whose near horizon geometry generically is not $AdS$ [15, 22, 4]. The superconformal description is recovered in the strong coupling limit of type-IIA decompactification, when $g^2_{eff} \gg N^{1/2}$, $u \ll g^2_{YM}$, where $g^2_{eff} = g_s N/u$ [4] and $u$ is the energy scale. When the radius of the circle goes to infinity, the gauge coupling also blows up and we reach the M2-brane description in the infrared of the three-dimensional gauge theory. Its relevant degrees of freedom are given in terms of the $d = 3$ chiral multiplets and are related to the KK excitations of $d = 11$ supergravity. Vice-versa, if $g^2_{eff} \to 0$ and $u \to \infty$, we are describing the ultraviolet limit of the $d = 3$ Yang-Mills theory (for a thorough discussion of the above considerations see [15, 22, 4]).

It is also important to observe that, although the $d = 3$ ultraviolet gauge theory has both a Coulomb and a Higgs branch, in the $AdS_4/CFT_3$ correspondence one is mainly interested in the Higgs branch, parametrized by the *vev*'s of the scalars of the chiral multiplets. The Coulomb branch scalars, belonging to the vector multiplets, are excluded since their *vev*'s can be safely put to zero.

In [23] by a thorough study of the KK excitations and OSp(4|2) multiplets [24] on the $M^{111} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\,\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ [25] and $Q^{111} = \mathrm{SU}(2)^3/\,\mathrm{U}(1)^2$ [26] manifolds, the relevant $\mathcal{N} = 2$ SCFT's have been constructed on the basis of the mass spectrum, and conjectured to be dual to $M$-theory compactifications.

In the above construction, it was important that both $M^{111}$ and $Q^{111}$ admit a description in terms of toric geometry [27]. This allowed to identify the fundamental degrees of freedom of the underlying SCFT and thus to find the abelian gauge theories, whose moduli space of vacua (the Higgs branch component) is isomorphic to the conifolds over the two seven-manifolds.

This paper analyses the case of M-theory compactified on the real Stiefel manifold $V_{(5,2)} \equiv \mathrm{SO}(5)/\,\mathrm{SO}(3)$ with SO(3) canonically embedded in SO(5). The relevant four-fold cone whose base is $V_{(5,2)}$ was identified in [5].

The Stiefel manifold is peculiar among the Einstein spaces leading to $\mathcal{N} = 2$ supersymmetry in that it does not admit a toric description [27]. Nevertheless, we shall see that it is possible to build up the *single brane* ($N = 1$) theory and to conjecture its $N > 1$ extension by performing the following steps.

We first analyse the full KK mass spectrum and OSp(2|4) supermultiplets for the $V_{(5,2)}$ manifold: this is in complete analogy with the procedure in [9] for $T^{11}$ and in [23, 25] for some M-theory examples. A relevant point of our analysis with respect to [23] is the absence of Betti multiplets since there are no non-trivial Betti numbers on $V_{(5,2)}$

$$b_p = 0\,, \qquad p = 1, \ldots, 6\,, \qquad b_0 = b_7 = 1\,. \tag{1.1}$$

Thus, there is no continuous baryonic symmetry [28, 29, 18, 9, 10] in the corresponding SCFT.

The second step is to construct admissible superconformal boundary operators to be put in dual correspondence with those KK supermultiplets undergoing shortening when the required unitarity bounds on the OSp(4|2) representations are saturated. To arrive to a well founded conjecture for the SCFT operators, we are guided by the consideration of the classical equation describing the eight dimensional cone $Y_8$ over $V_{(5,2)}$. It turns out that the solution of the cone equation [5]

$$\sum_{a=1}^{4} z_a^2 = 0 \tag{1.2}$$

can be obtained in the simplest way by use of the so called Plücker embedding, where beside the pfaffian identity on the coordinates $p_{ij} = -p_{ji}$ (where $i, j$ are indices in the fundamental representation of $\mathrm{Sp}(4) \equiv \mathrm{SO}(5)$)

$$\epsilon^{ijkl} p_{ij} p_{kl} = 0 \tag{1.3}$$

one also uses the traceless constraint

$$C^{ij} p_{ij} = 0\,. \tag{1.4}$$

As shown in [23], the solution to (1.3) can be given as

$$p_{ij} = A_{[i} B_{j]}\,. \tag{1.5}$$

In the present case, we find that one can write the coordinates parametrizing the cone $Y_8$ as the bilinear

$$z^a = A^i\ \Gamma^a_{ij} B^j\,, \tag{1.6}$$

where $A^i$, $B^i$ are in the $\mathbf{4}$ of $\mathrm{Sp}(4) \equiv \mathrm{SO}(5)$ and $\Gamma^a_{ij}$ are the gamma matrices in five dimensions. The vanishing of a SU(2) D-term fixes the residual $\mathrm{SL}(2, \mathbb{C})$ invariance of equations (1.3) and (1.4). The above solution (1.6) is obtained by a procedure quite analogous to that employed in [5] to solve the cone equation on $T^{11}$ in terms of two objects $A^i$, $B^i$ belonging to the representation $(1/2, 0) \oplus (0, 1/2)$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. There, the analogous of $\mathrm{SL}(2, \mathbb{C})$ invariance was given by [5, eq. (13)], namely invariance under complex rescalings.



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This discussion gives us a little but useful information on the gauge group $\mathcal{G}$ of the theory in the ultraviolet regime. More precisely, since the equation for the vanishing of the $D$-terms is SU(2) valued, the gauge group should reduce to SU(2) for a single $M2$-brane at the conical singularity. Then, in virtue of the fact that the conifold coordinate $z^a$ does indeed appear in the KK spectrum and is a gauge singlet, we arrive at the conclusion that for $N > 1$ the basic singleton $S_i^\alpha$, transforming in the **4** of SO(5), must be in a pseudoreal representation of $\mathcal{G}$ labeled by the index $\alpha$. Albeit this requirements could be satisfied by several groups, we are led to conjecture the simple choice $\mathcal{G} = \mathrm{USp}(2N) \times \mathrm{O}(2N-1)$, where the index $\alpha$ of $S_i^\alpha$ belongs to the bifundamental representation[1] of $\mathcal{G}$.

Once we have some solid base for the choice of the basic degrees of freedom in the dual three-dimensional $\mathcal{N} = 2$ SCFT, we can perform the last step and show that it is possible to construct a complete set of conformal primary operators *matching all* the KK multiplets previously obtained.

In establishing such correspondence we follow the procedure already employed in [9] for the $T^{11}$ case. In particular we find that, as for $T^{11}$, there are long multiplets with rational protected dimensions.[2]

It is interesting to note that since in the infrared limit the gauge field is integrated out, one may expect that it should be related in the SCFT to some composite field in terms of the singleton $S_i^\alpha$. In fact we find in the gravitino sector a composite field $X_\alpha$ obeying $\bar{D}^\alpha X_\alpha = 0$ whose $\bar{\theta}$ component[3] has the right quantum numbers of a gauge field, being a singlet of the flavour group SO(5) and having $R$-symmetry $y = 0$.

In the rest of the paper, section 2 briefly deals with the harmonic analysis on the Stiefel manifold while section 3 contains the results on the full mass spectrum and its assembling into OSp(4|2) supermultiplets, with particular emphasis on the shortening patterns due to saturation of unitarity bounds.

Section 4, relying on the solution of the conifold equation, proposes a candidate for the classical $N = 1$ theory which is supported by the condition of vanishing D-terms. The $N > 1$ extension is then conjectured to be related in the ultraviolet to a gauge theory of $D2$-branes given by the product of two non-simply laced groups $\mathcal{G} = \mathrm{USp}(2N) \times \mathrm{O}(2N-1)$ with chiral multiplet $S_i^\alpha$ transforming in the spinor representation of the flavour group SO(5) and in the bifundamental of $\mathcal{G}$.

Some considerations are also given on the possible existence of a superpotential, at least in the $N = 1$ case. In section 5, after a short summary of the conformal operators related to the shortenings of KK representations, we construct a set of

[1] Indeed such gauge group appear in the context of orientifold models [30].

[2] The same feature was also found in [23].

[3] For three-dimensional superfields we define: $\theta_\alpha = \theta^1_\alpha + i\theta^2_\alpha$, $\bar{\theta}_\alpha = \theta^1_\alpha - i\theta^2_\alpha$. Conformal dimensions $\Delta$ and $R$-symmetry $y$ quantum numbers are $(\Delta = 1/2, y = 1)$ for $\theta_\alpha$ and $(\Delta = 1/2, y = -1)$ for $\bar{\theta}_\alpha$.

conformal operators which can be put in correspondence with the various supermultiplets. Finally, we give a summary of our result in section 6 while some of the more technical aspects regarding useful tools for the harmonic analysis are contained in appendices A and B.

## 2. The mass spectrum of $V_{(5,2)}$

Harmonic analysis on the coset space $V_{(5,2)} \equiv \mathrm{SO}(5)/\,\mathrm{SO}(3)$ yields the complete mass spectrum, which as in the other $\mathcal{N}=2$ supersymmetric compactifications, is arranged into $\mathrm{OSp}(4|2)$ representations. Referring to [10] and references therein for the relevant details concerning harmonic expansion on a generic coset manifold, we give below the essential ingredients for carrying out the computations and collect our results in tables for the various supermultiplets.

As in any KK compactification, after the linearization of the equations of motion of the field fluctuations, one is left with a differential equation on the eleven-dimensional fields $\Phi^{\{J\}}_{[\lambda_1,\lambda_2,\lambda_3]}(x,y)$

$$\left(\Box_x^{\{J\}} + \boxtimes_y^{[\lambda_1,\lambda_2,\lambda_3]}\right)\Phi^{\{J\}}_{[\lambda_1,\lambda_2,\lambda_3]}(x,y) = 0\,. \tag{2.1}$$

Here the field $\Phi^{\{J\}}_{[\lambda_1,\lambda_2,\lambda_3]}(x,y)$ depends on the coordinates $x$ of $AdS_4$ and $y$ of $V_{(5,2)}$, and transforms irreducibly in the representations $\{J\}$ of $\mathrm{SO}(3,2)$ and $[\lambda_1,\lambda_2,\lambda_3]$ of $\mathrm{SO}(7)$. $\Box_x$ is the kinetic operator for a field of quantum numbers $\{J\}\equiv\{\Delta,s\}$ in four-dimensional $AdS$ space and $\boxtimes_y$ is the Laplace-Beltrami operator for a field of spin $[\lambda_1,\lambda_2,\lambda_3]$ in the internal space $V_{(5,2)}$. (In the following we mostly omit the index $\{J\}$ on the fields. The symbol $[\lambda_1,\lambda_2,\lambda_3]$ denotes the quantum numbers of the $\mathrm{SO}(7)$ representation in the Young tableaux formalism.)

More specifically, one expands the $d=11$ supergravity fields $\Phi_{[\lambda_1,\lambda_2,\lambda_3]}(x,y) = \{h_{\hat a\hat b},\ A_{\hat a\hat b\hat c}, \psi_{\hat a}\}$ ($\hat a=(a,\alpha)$, $a=1,\ldots,4$, $\alpha=1,\ldots,7$) in the harmonics of $V_{(5,2)}$ transforming irreducibly under the isometry group of $V_{(5,2)}$, and computes the action of $\boxtimes_y$ on these harmonics. The eigenvalues are simply related to the $AdS$ mass.

The two necessary ingredients in this computation are the geometric structure and the harmonics of the coset space.

**Geometry.** We give here a brief description of the essential geometrical elements of the Stiefel manifold such as the metric and the riemmanian curvature, that are used to build the invariant Laplace-Beltrami operators.

We remind that the Stiefel manifold, beside $\mathrm{SO}(5)$, has an extra isometry $\mathrm{SO}(2)_R$ that can be identifyed with the $R$-symmetry group [19]

$$\frac{G}{H} = V_{(5,2)} \equiv \frac{\mathrm{SO}(5)\times\mathrm{SO}(2)_R}{\mathrm{SO}(3)\times\mathrm{SO}(2)_H}\,, \tag{2.2}$$

where the embedding of the $SO(2)_H$ into $SO(5) \times SO(2)_R$ is diagonal and the embedding of SO(3) in SO(5) is the canonical one, namely the fundamental of SO(5) breaks under SO(3) as $\mathbf{5} \to \mathbf{3} + \mathbf{1} + \mathbf{1}$ (other embeddings yield different inequivalent $M_7$ with $\mathcal{N} \neq 2$).

We call $\Lambda, \Sigma = 1, \ldots, 5$ the SO(5) indices, $I = 1, 2, 3$ the SO(3) indices and $A = 1, 2$ the SO(2) ones. The adjoint generators of SO(5) and $SO(2)_R \simeq U(1)_R$ are respectively $T^{\Lambda\Sigma}, U$.

The SO(5) algebra is

$$[T_{\Lambda\Sigma}, T_{\Gamma\Delta}] = -\eta_{\Lambda\Gamma} T_{\Sigma\Delta} + \eta_{\Lambda\Delta} T_{\Sigma\Gamma} - \eta_{\Sigma\Delta} T_{\Lambda\Gamma} + \eta_{\Sigma\Gamma} T_{\Lambda\Delta} \,, \tag{2.3}$$

which means that our generators in the vector representation are $(T^{\Lambda\Sigma})_{\Gamma\Delta} = 2\delta^{\Lambda\Sigma}_{\Gamma\Delta}$.

For the canonical embedding of SO(3) in SO(5), $\Lambda = (I, A)$, we can define $J_K \equiv \frac{1}{2}\epsilon_{KIJ} T_{IJ}$ as the SO(3) generators and $N \equiv T_{45} + U$ as the $SO(2)_H$ generator. The coset generators are given by $T_{IA} \equiv (T_m, T_{\hat{m}})$, $(m, \hat{m} = 1, 2, 3)$, $T_7 \equiv T_{45} - U$.

Since the vielbeins are coset-algebra valued, we use the same convention for labelling the vielbein directions in the coset space.

Given the structure constants $C^a{}_{bc}$ of the coset, the Riemann tensor is defined by the formula

$$\begin{aligned} R^a{}_{bde} = & -\frac{1}{4}\mathbb{C}^a{}_{bc} C^c{}_{de} \frac{r(d) r(e)}{r(c)} - \frac{1}{2} C^a{}_{bi} C^i{}_{de} r(d) r(e) - \\ & -\frac{1}{8}\mathbb{C}^a{}_{cd} \mathbb{C}^c{}_{be} + \frac{1}{8}\mathbb{C}^a{}_{ce} \mathbb{C}^c{}_{bd} \,, \end{aligned} \tag{2.4}$$

whose derivation we give in appendix A. Here we simply point out that the $r(a)$ are the rescalings of the vielbeins needed to obtain an Einstein space and that the $\mathbb{C}^a{}_{bc}$ are certain specific combinations of the structure constants.

We have imposed[4]

$$r(a) = r(b) = 4\sqrt{\frac{2}{3}}\, e, \qquad r(c) = -\frac{4}{3}\, e \tag{2.5}$$

to obtain $R^a{}_b = 12 e^2\, \delta^a_b$. With such rescalings (2.5), we obtain

$$\begin{aligned} R^{mn}{}_{kl} &= R^{\hat{m}\hat{n}}{}_{\hat{k}\hat{l}} = \frac{32}{3}\, \delta^{mn}_{kl} \\ R^{mn}{}_{\hat{k}\hat{l}} &= \frac{20}{3} \delta^{mn}_{kl} \\ R^{m\hat{n}}{}_{k\hat{l}} &= \frac{4}{3} \delta^{mn} \delta_{kl} - 2\delta^m_l \delta^n_k \\ R^{7m}{}_{7n} &= R^{7\hat{m}}{}_{7\hat{n}} = 2\delta^m_n \,. \end{aligned} \tag{2.6}$$

[4]Note that there is an ambiguity in the sign of the rescalings, since the Einstein space requirement on the curvature determines only their square. However, this ambiguity is only apparent. While the partially reflected solutions with $r(a) \to -r(a)$ or $r(b) \to -r(b)$ are perfectly equivalent to our description, a change in the sign of $r(c)$ implies that we really reflect the orientation on the manifold and as a consequence we completely break supersymmetry.



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**The harmonics.** The harmonics on the coset space $V_{(5,2)}$ are labelled by two kind of indices, the first giving the specific representation of the isometry group $\mathrm{SO}(5) \times \mathrm{U}_R(1)$ and the other referring to the representation of the subgroup $H \equiv \mathrm{SO}(3) \times \mathrm{SO}(2)$. The harmonic is thus denoted by $Y^{(M,N,Q)}_{(q_H)}(y)$, where $M$,$N$ are the quantum numbers of the SO(5) representation, $Q$ is the $\mathrm{U}_R(1)$ charge and $q_H$ are the $H$-quantum numbers.

The above results imply that an SO(7) field $\Phi_{[\lambda_1,\lambda_2,\lambda_3]}(x,y)$ can be splitted into the direct sum of $H$ irreducible fragments labelled by $q_H$. The analysis of the reduction of the SO(7) representation under the $H$ group reported in appendix B, yields that the vector and spinor SO(7) representations break as

$$\begin{aligned} &\mathbf{7} \to \mathbf{3}_1 \oplus \mathbf{3}_{-1} \oplus \mathbf{1}_0\,, \\ &\mathbf{8} \to \mathbf{3}_{1/2} \oplus \mathbf{3}_{-1/2} \oplus \mathbf{1}_{3/2} \oplus \mathbf{1}_{-3/2}\,, \end{aligned} \tag{2.7}$$

and by taking suitable combinations one can also derive all the other tensor decompositions.

The generic field $\Phi_{[\lambda_1,\lambda_2,\lambda_3]}(x,y)$ can then be expanded as follows

$$\Phi_{ab\ldots}(x,y) = \sum_{(\nu)} \sum_{(m)} \Phi_{(\nu)(m)}(x) Y^{(\nu)(m)}_{ab\ldots}(y)\,, \tag{2.8}$$

where $a, b, \ldots$ are SO(7) tensor (or spinor) indices of the representation $[\lambda_1, \lambda_2, \lambda_3]$, $(\nu)$ is a shorthand notation for $(M,N,Q)$ and $m$ labels the representation space of $(M,N,Q)$.

Of course, not all the harmonics are allowed in the (2.8) expansion, as the irrepses of SO(7) appearing in (2.8) must contain, once reduced with respect to $H$, at least one of the representations appearing in the decomposition of $[\lambda_1, \lambda_2, \lambda_3]$ under $H$. This gives some constraints on $M, N, Q$ which select the allowed representations $(\nu)$.

We write a generic representation of SO(5) in the Young tableaux formalism $[\lambda_1, \lambda_2] = [M+N, M]$:

$$\begin{array}{l} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \phantom{x} & \phantom{x} & \cdots & \phantom{x} & \phantom{x} & \phantom{x} & \cdots & \phantom{x} \\ \hline \phantom{x} & \phantom{x} & \cdots & \phantom{x} & \multicolumn{4}{c}{} \\ \cline{1-4} \end{array} \\ \underbrace{\phantom{xxxxxxxxxx}}_{M}\underbrace{\phantom{xxxxxxxxxx}}_{N} \end{array} \tag{2.9}$$

and $Q$, the $\mathrm{U}_R(1)$ charge, is defined by the $\mathrm{U}_R(1)$ harmonic $e^{iQ\phi}$.

A specific component of (2.9) can then be written as

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 4 & 4 & \ldots & i & j & + & \ldots & - \\ \hline 5 & 5 & \ldots & + & \multicolumn{4}{c}{} \\ \cline{1-4} \end{array}$$

where we defined the $\mathrm{U}_R(1)$ fixed charge combinations

$$\boxed{\ \pm\ } \equiv \boxed{\ 4\ } \pm i \boxed{\ 5\ }$$



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| spin | $\Phi(x)$ | $\Phi(x,y)$ | harmonic | $\boxtimes$ operator | SO(7) irrep |
|---|---|---|---|---|---|
| 2 | $h_{ab}$ | $h_{ab}$ | Y | $\Box = \mathcal{D}^{\alpha}\mathcal{D}_{\alpha}$ | $[0,0,0]$ |
| $1^+$ | $A_a$, $W_a$ | $h_{a\beta}$, $A_{ab\gamma}$ | $Y_{\alpha}$ | $\Box + 24$ | $[1,0,0]$ |
| $1^-$ | $Z_a$ | $A_{a\beta\gamma}$ | $Y_{[\beta\gamma]}$ | $(\Box+40)\delta^{\mu\nu}_{\beta\gamma} - 2\mathcal{C}^{\mu\nu\beta\gamma}$ | $[1,1,0]$ |
| $0^+$ | S,$\Sigma$ | $h_{ab}$, $h_{\alpha\beta}$, $A_{abc}$ | Y | $\Box$ | $[0,0,0]$ |
| | $\phi$ | $h_{\alpha\beta}$ | $Y_{(\alpha\beta)}$ | $(\Box+40)\delta^{(\mu\nu)}_{(\beta\gamma)} - 4\mathcal{C}^{\mu\beta\nu\gamma}$ | $[2,0,0]$ |
| $0^-$ | $\pi$ | $A_{\alpha\beta\gamma}$ | $Y_{[\alpha\beta\gamma]}$ | $1/24\epsilon^{\mu\nu\rho\sigma\alpha\beta\gamma}\mathcal{D}_{\sigma}$ | $[1,1,1]$ |
| 3/2 | $\chi_a$ | $\psi_a$ | $\Xi$ | $\not{\mathcal{D}} - 7$ | $[\frac{1}{2},\frac{1}{2},\frac{1}{2}]$ |
| 1/2 | $\lambda_L$ | $\psi_a$,$\psi_\alpha$ | $\Xi$, $\Xi_\alpha$ | | |
| | $\lambda_T$ | $\psi_\alpha$ | $\Xi_\alpha$ | $\not{\mathcal{D}} - 5$ | $[\frac{3}{2},\frac{1}{2},\frac{1}{2}]$ |

**Table 1:** Correspondence between 11d and 4d fields and harmonics [31].

One can now proceed with the KK analysis, implementing the above informations in all the Laplace-Beltrami operators and computing the eigenvalues of the various harmonics and thus the masses of the KK states.

Although straightforward in principle, this analysis can become quite cumbersome for some of the higher spin operators. Luckily, it is not really necessary to complete the whole task. In fact, due to the $\mathcal{N}=2$ supersymmetry, this peculiar compactification falls in the class considered in [24], where the OSp(4|2) multiplet structure was elucidated, together with the mass values expected for states of given quantum numbers. Most of these multiplets can be filled by using only our results for the simpler operators, while the entries in the remaining slots can be determined with the help of supersymmetry.

From the scalar, spinor and vector eigenvalues, we have obtained the masses for all the graviton and gravitinos and for some of the vectors, spinors and scalars, which let us fill the five types of supermultiplets presented in tables 2–6, with all the shortening patterns. A preliminary analysis of the rank of the two-form matrix yields that we can have at most two more vector multiplets, which we guess do not undergo shortening.

Indeed, as in the $T^{11}$ case [10, 9], all the mass eigenvalues depend on the $G$-quantum numbers only through the function $H_0$, which is the scalar laplacian eigenvalue. For the Stiefel manifold, such eigenvalue is given by

$$H_0(M,N,Q) = \frac{32}{9}\left(6M^2 + 9N + 3N^2 + 12M + 6MN - Q^2\right). \tag{2.10}$$

Since for a given number of preserved supersymmetries, the structure of the linearized equations, supersymmetry relations and supermultiplets are fixed, we can suppose that also the mass formulae in terms of $H_0$ are universal for all seven-dimensional $\mathcal{N}=2$ supersymmetric compactifications. By this we mean that not only the number and type of multiplets for different compactifications are the same, but also the $H_0$ mass dependence should be equal.

This is exactly what we find by comparing our case with the $M^{111}$ compactification [25], and we expect such agreement to hold also for the $Q^{111}$ space. Of course, the shortening patterns as well as the possible presence of Betti multiplets will be model dependent features, as they derive either from certain functions of $H_0$ taking rational values or from non-trivial Betti numbers of the relevant manifold. However, in this respect, the two vector multiplets mentioned before have $\Delta = \frac{1}{2} + \frac{1}{4}\sqrt{4 + H_0}$ and $\Delta = \frac{1}{2}+\frac{1}{4}\sqrt{H_0(M, N, Q + \frac{3}{2}) - 28}$, and satisfy no shortening conditions. This makes us confident that *all* the relevant output derived from the supergravity analysis is correct.

## 3. The $AdS_4 \times V_{(5,2)}$ multiplet structure

We report below in tables 2–6 the five families of supermultiplets we have found: one graviton multiplet, two gravitino multiplets and two vector multiplets.

Each table has five main columns. The first column contains the spin quantum number of the state, while in the second we give its $\Delta^{(s)}$ value. The basic value of $\Delta$ assigned to each multiplet is the one belonging to a vector field, a spin 1/2 or a scalar for the graviton, gravitino and vector multiplets respectively. In the third column we write the $R$-symmetry of the state where the value $y$ is assigned to the state with $\Delta^{(s)} = \Delta$. We use here $y = \frac{2}{3}Q$, since this varies in integer steps according with the usual convention on the unit value of the $R$-charge of the $\theta$ coordinate [24]. The fourth column shows the specific field of the KK spectrum that is associated with the given OSp(4|2) state, according to the notations of [24]. The fifth column contains the mass of the state[5] given in terms of $H_0$.

For generic SO(5) quantum numbers and $R$-symmetry values, the multiplets of tables 2–6 are long multiplets of OSp(4|2). However, group theory predicts [24] shortening in correspondence with specific threshold values of the quantum numbers. These give rise to chiral ($\bullet$), semi-long ($\star$) or massless ($\diamond$) multiplets. The above symbols appear in the extra left columns to denote the surviving states in the shortened multiplets. In absence of these symbols no shortening of any kind can occur for that multiplet.

## 4. Classical $V_{(5,2)}$ cone equation and CFT

Consider the non-compact four-fold defined by

$$\sum_{a=1}^{5} z_a^2 = 0\,, \tag{4.1}$$

which has an ordinary double point singularity at $z_a = 0$ [5]. This conifold is a cone over the homogeneous space SO(5)/SO(3), that can be retrieved by looking at the

[5] We give the value of the mass for the fermions and the mass squared for the bosons.

| | | spin | $\Delta^{(s)}$ | $R$-symm. | field | Mass |
|---|---|---|---|---|---|---|
| $\diamond$ | $\star$ | 2 | $\Delta+1$ | $y$ | $h$ | $H_0$ |
| $\diamond$ | $\star$ | 3/2 | $\Delta+1/2$ | $y-1$ | $\chi^+$ | $-6+\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | 3/2 | $\Delta+1/2$ | $y+1$ | $\chi^+$ | $-6+\sqrt{H_0+36}$ |
| | $\star$ | 3/2 | $\Delta+3/2$ | $y-1$ | $\chi^-$ | $-6-\sqrt{H_0+36}$ |
| | | 3/2 | $\Delta+3/2$ | $y+1$ | $\chi^-$ | $-6-\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | 1 | $\Delta$ | $y$ | $A/W$ | $H_0+48-8\sqrt{H_0+36}$ |
| | | 1 | $\Delta+1$ | $y+2$ | $Z$ | $H_0+32$ |
| | $\star$ | 1 | $\Delta+1$ | $y-2$ | $Z$ | $H_0+32$ |
| | $\star$ | 1 | $\Delta+1$ | $y$ | $Z$ | $H_0+32$ |
| | | 1 | $\Delta+1$ | $y$ | $Z$ | $H_0+32$ |
| | | 1 | $\Delta+2$ | $y$ | $A/W$ | $H_0+48+8\sqrt{H_0+36}$ |
| | | 1/2 | $\Delta+1/2$ | $y+1$ | $\lambda_T^-$ | $2-\sqrt{H_0+36}$ |
| | $\star$ | 1/2 | $\Delta+1/2$ | $y-1$ | $\lambda_T^-$ | $2-\sqrt{H_0+36}$ |
| | | 1/2 | $\Delta+3/2$ | $y-1$ | $\lambda_T^+$ | $2+\sqrt{H_0+36}$ |
| | | 1/2 | $\Delta+3/2$ | $y+1$ | $\lambda_T^+$ | $2+\sqrt{H_0+36}$ |
| | | 0 | $\Delta+1$ | $y$ | $\phi$ | $H_0+32$ |

**Table 2:** Long graviton multiplet. $\Delta=\frac{1}{2}+\frac{1}{4}\sqrt{H_0+36}$.

| | spin | $\Delta^{(s)}$ | $R$-symm. | field | Mass |
|---|---|---|---|---|---|
| $\star$ | 3/2 | $\Delta+1$ | $y$ | $\chi^+$ | $-8+\sqrt{H_0+24}$ |
| $\star$ | 1 | $\Delta+1/2$ | $y+1$ | $A/W$ | $H_0+56-12\sqrt{H_0+24}$ |
| $\star$ | 1 | $\Delta+1/2$ | $y-1$ | $A/W$ | $H_0+56-12\sqrt{H_0+24}$ |
| $\star$ | 1 | $\Delta+3/2$ | $y-1$ | $Z$ | $H_0+24+4\sqrt{H_0+24}$ |
| | 1 | $\Delta+3/2$ | $y+1$ | $Z$ | $H_0+24+4\sqrt{H_0+24}$ |
| $\star$ | 1/2 | $\Delta$ | $y$ | $\lambda_T^+$ | $-8-\sqrt{H_0+24}$ |
| $\star$ | 1/2 | $\Delta+1$ | $y-2$ | $\lambda_T^-$ | $4+\sqrt{H_0+24}$ |
| $\star$ | 1/2 | $\Delta+1$ | $y$ | $\lambda_T^-$ | $4+\sqrt{H_0+24}$ |
| | 1/2 | $\Delta+1$ | $y+2$ | $\lambda_T^-$ | $4+\sqrt{H_0+24}$ |
| | 1/2 | $\Delta+1$ | $y$ | $\lambda_T^-$ | $4+\sqrt{H_0+24}$ |
| | 1/2 | $\Delta+2$ | $y$ | $\lambda_T^+$ | $-\sqrt{H_0+24}$ |
| $\star$ | 0 | $\Delta+1/2$ | $y-1$ | $\pi$ | $H_0+56-12\sqrt{H_0+24}$ |
| | 0 | $\Delta+1/2$ | $y+1$ | $\pi$ | $H_0+56-12\sqrt{H_0+24}$ |
| | 0 | $\Delta+3/2$ | $y+1$ | $\phi$ | $H_0+24+4\sqrt{H_0+24}$ |
| | 0 | $\Delta+3/2$ | $y-1$ | $\phi$ | $H_0+24+4\sqrt{H_0+24}$ |

**Table 3:** Long gravitino multiplet I. $\Delta=-\frac{1}{2}+\frac{1}{4}\sqrt{H_0+24}$.



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| | spin | $\Delta^{(s)}$ | $R$-symm. | field | Mass |
|---|---|---|---|---|---|
| $\star$ | $3/2$ | $\Delta+1$ | $y$ | $\chi^-$ | $-8+\sqrt{H_0+24}$ |
| $\star$ | $1$ | $\Delta+1/2$ | $y+1$ | $Z$ | $H_0+24-4\sqrt{H_0+24}$ |
| $\star$ | $1$ | $\Delta+1/2$ | $y-1$ | $Z$ | $H_0+24-4\sqrt{H_0+24}$ |
| $\star$ | $1$ | $\Delta+3/2$ | $y-1$ | $A/W$ | $H_0+56+12\sqrt{H_0+24}$ |
| | $1$ | $\Delta+3/2$ | $y-1$ | $A/W$ | $H_0+56+12\sqrt{H_0+24}$ |
| $\star$ | $1/2$ | $\Delta$ | $y$ | $\lambda_T^-$ | $+\sqrt{H_0+24}$ |
| $\star$ | $1/2$ | $\Delta+1$ | $y-2$ | $\lambda_T^+$ | $4-\sqrt{H_0+24}$ |
| $\star$ | $1/2$ | $\Delta+1$ | $y$ | $\lambda_T^+$ | $4-\sqrt{H_0+24}$ |
| | $1/2$ | $\Delta+1$ | $y+2$ | $\lambda_T^+$ | $4-\sqrt{H_0+24}$ |
| | $1/2$ | $\Delta+1$ | $y$ | $\lambda_T^+$ | $4-\sqrt{H_0+24}$ |
| | $1/2$ | $\Delta+2$ | $y$ | $\lambda_T^-$ | $-8+\sqrt{H_0+24}$ |
| $\star$ | $0$ | $\Delta+1/2$ | $y-1$ | $\phi$ | $H_0+24-4\sqrt{H_0+24}$ |
| | $0$ | $\Delta+1/2$ | $y+1$ | $\phi$ | $H_0+24-4\sqrt{H_0+24}$ |
| | $0$ | $\Delta+3/2$ | $y+1$ | $\pi$ | $H_0+56-12\sqrt{H_0+24}$ |
| | $0$ | $\Delta+3/2$ | $y-1$ | $\pi$ | $H_0+56-12\sqrt{H_0+24}$ |

**Table 4:** Long gravitino multiplet II. $\Delta=\frac{3}{2}+\frac{1}{4}\sqrt{H_0+24}$.

| spin | $\Delta^{(s)}$ | $R$-symm. | field | Mass |
|---|---|---|---|---|
| $1$ | $\Delta+1$ | $y$ | $A/W$ | $H_0+96+16\sqrt{H_0+36}$ |
| $1/2$ | $\Delta+1/2$ | $y-1$ | $\lambda_T$ | $6+\sqrt{H_0+36}$ |
| $1/2$ | $\Delta+1/2$ | $y+1$ | $\lambda_T$ | $6+\sqrt{H_0+36}$ |
| $1/2$ | $\Delta+3/2$ | $y-1$ | $\lambda_L$ | $10+\sqrt{H_0+36}$ |
| $1/2$ | $\Delta+3/2$ | $y+1$ | $\lambda_L$ | $10+\sqrt{H_0+36}$ |
| $0$ | $\Delta$ | $y$ | $\phi$ | $24+H_0+8\sqrt{H_0+36}$ |
| $0$ | $\Delta+1$ | $y-2$ | $\pi$ | $H_0+96+16\sqrt{H_0+36}$ |
| $0$ | $\Delta+1$ | $y$ | $\pi$ | $H_0+96+16\sqrt{H_0+36}$ |
| $0$ | $\Delta+1$ | $y+2$ | $\pi$ | $H_0+96+16\sqrt{H_0+36}$ |
| $0$ | $\Delta+2$ | $y$ | $S/\Sigma$ | $176+H_0+24\sqrt{H_0+36}$ |

**Table 5:** Vector multiplet I. $\Delta=\frac{5}{2}+\frac{1}{4}\sqrt{H_0+36}$.

set of points at unit distance from the singularity

$$\sum_{a=1}^{5}|z_a|^2=1\,. \tag{4.2}$$

The full isometry group of this space is $\mathrm{SO}(5)\times\mathrm{U}_R(1)$ where the $\mathrm{U}_R(1)$ plays the role of an $R$-symmetry group and acts as a phase shift on the coordinates

$$z_a\to e^{i\alpha}z_a\,. \tag{4.3}$$

Therefore the $z_a$ have $Q=1$ under this symmetry and transform in the **5** of SO(5).

| | | | spin | $\Delta^{(s)}$ | $R$-symm. | field | Mass |
|---|---|---|---|---|---|---|---|
| $\diamond$ | $\star$ | | 1 | $\Delta+1$ | $y$ | $A/W$ | $H_0+96-16\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | $\bullet$ | 1/2 | $\Delta+1/2$ | $y-1$ | $\lambda_L$ | $10-\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | | 1/2 | $\Delta+1/2$ | $y+1$ | $\lambda_L$ | $10-\sqrt{H_0+36}$ |
| | $\star$ | | 1/2 | $\Delta+3/2$ | $y-1$ | $\lambda_T$ | $6-\sqrt{H_0+36}$ |
| | | | 1/2 | $\Delta+3/2$ | $y+1$ | $\lambda_T$ | $6-\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | $\bullet$ | 0 | $\Delta$ | $y$ | $S/\Sigma$ | $176+H_0-24\sqrt{H_0+36}$ |
| | $\star$ | $\bullet$ | 0 | $\Delta+1$ | $y-2$ | $\pi$ | $H_0+96-16\sqrt{H_0+36}$ |
| $\diamond$ | $\star$ | | 0 | $\Delta+1$ | $y$ | $\pi$ | $H_0+96-16\sqrt{H_0+36}$ |
| | | | 0 | $\Delta+1$ | $y+2$ | $\pi$ | $H_0+96-16\sqrt{H_0+36}$ |
| | | | 0 | $\Delta+2$ | $y$ | $\phi$ | $24+H_0-8\sqrt{H_0+36}$ |

**Table 6:** Vector multiplet II. $\Delta=-\frac{3}{2}+\frac{1}{4}\sqrt{H_0+36}$.

Since it acts non-trivially on the canonical line bundle of the conifold, the (4.3) transformation is an $R$-symmetry of the theory. We can also see that it is an $R$-symmetry group from the fact that the holomorphic 4-form

$$\Omega=\frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_5} \tag{4.4}$$

has $Q=3$ under the $\mathrm{U}_R(1)$ $(\Omega\to e^{3i\alpha}\Omega)$. The charge of the fermionic coordinates of superspace is fixed by the requirement that they transform as $\sqrt{\Omega}$, and then $Q_\theta=3/2$. Indeed, on a Calabi-Yau manifold we can always write the holomorphic form as

$$\Omega_{abcd}={}^t\eta\Gamma_{abcd}\eta\,, \tag{4.5}$$

where $\eta$ is a covariantly constant spinor. This means that $\Omega$ transforms as $\eta^2$, and supersymmetries, being generated by covariantly constant spinors, transform as $\eta$.

As explained in section 3, it is convenient to fix the $R$-symmetry value of the $\theta$ coordinates equal to one, and to introduce the rescaled $R$-charge $y=\frac{2}{3}Q$, under which $y_\theta=1$.

In complete analogy with [5], we can write a CY metric on the cone by introducing the SO(5) invariant Kähler potential

$$\mathcal{K}=\left(\sum_a \bar{z}_a z_a\right)^{3/4}. \tag{4.6}$$

Defining $r\equiv\left(\sum_a \bar{z}_a z_a\right)^{3/8}$ and introducing a set of angular variables $y^A$, invariant under the scaling of the $z$ coordinates, the metric can be put in the standard form

$$ds^2_C=dr^2+r^2g_{AB}dy^Ady^B\,,\qquad (A,B=1,\ldots,7)\,. \tag{4.7}$$

This metric inserted in

$$ds^2_{11} = r^2 \left(dx_0^2 - dx_1^2 - dx_2^2\right) + \frac{1}{r^2}\, ds^2_C \tag{4.8}$$

plus the vacuum expectation value of the three-form field strength ($F_4 \equiv dA_3$)

$$F_{\mu\nu\rho\sigma} = e\, \epsilon_{\mu\nu\rho\sigma} \tag{4.9}$$

describe the supergravity vacuum yielding the spontaneous compactification on a seven-manifold from eleven to four space-time dimensions.

This supergravity solution has no moduli, as in eleven dimensional supergravity there is no dilaton and the vev's of the fields giving the $AdS_4 \times V_{(5,2)}$ compactification is uniquely fixed. The only "$\theta$-angle" we could introduce is a shift in the vacuum value of the three-form $A_{\hat{a}\hat{b}\hat{c}}$ by a closed non-exact three-form on the internal indices. But we know that $H_3(V_{(5,2)}, \mathbb{Z})$ is at most discrete torsion [23] and therefore there are no "$\theta$-angles". The absence of moduli reflects in the CFT definition implying that the interacting fixed point is isolated in the parameter space.

This seems to be related to the geometrical nature of this manifold. It has been shown [27] that, at variance with the $M^{111}$ and $Q^{111}$ cases, the Stiefel manifold does not admit a description in terms of toric geometry and thus it is very difficult to see if it can be found as a partial resolution of some orbifold. If this could be done (like for the $Q^{111}$ manifold [27]), it would imply that there exists a flux from the orbifold CFT to this infrared point [5, 18], but it does not seem to be the case. Recent supergravity calculations [32] seem to confirm this fact at least for fluxes connecting manifolds with the same topology.

**The conformal field theory.** In the same spirit of [5], the basic degrees of freedom of the desired CFT can be understood upon "solving" the (4.1) equation. This can be done as follows: we set

$$z^a = {}^tA\ \Gamma^a B \equiv {}^tA^i\ \Gamma^a_{ij} B^j\,, \tag{4.10}$$

where $A^i$ and $B^i$ are SO(5) spinors (transforming in the fundamental representation of Sp(4)) and $\Gamma^a$ are antisymmetric gamma matrices in five dimensions, namely

$$\Gamma^a_{ij} = C_{ik} (\Gamma^a)^k{}_j\,, \tag{4.11}$$

$C_{ik}$ being the Sp(4) invariant metric. Since (using the identity $\Gamma^a_{ij}\Gamma_{a\,kl} = -6C_{[ij}C_{k]l} + C_{ij}C_{kl}$)

$$\sum_{a=1}^{5} z^a z_a \equiv ({}^tA\ \Gamma^a B)({}^tA\ \Gamma_a B) \sim ({}^tA\ CB)^2\,, \tag{4.12}$$

we have to supplement (4.10) with the symplectic trace condition

$$C_{ij} A^i B^j = 0 \tag{4.13}$$

in order to retrieve the conifold equation (4.1).

This matches exactly the representation of the conifold already used in [23] in terms of the Plücker coordinates

$$p_{ij} = A_{[i}B_{j]}\,, \tag{4.14}$$

satisfying the pfaffian constraint

$$C^{[ij}C^{kl]}p_{ij}p_{kl} = 0\,, \tag{4.15}$$

supplemented with the traceless condition $C^{ij}p_{ij} = 0$.[6] Equations (4.13) and (4.14) are invariant under $\mathrm{SL}(2,\mathbb{C})$ transformations. If we set $A^i = S^i_1$ and $B^i = S^i_2$, we see that the Plücker coordinates

$$p_{ij} = A_{[i}B_{j]} \equiv S^i_\alpha S^j_\beta \epsilon^{\alpha\beta}\,, \qquad \alpha,\beta = 1,2 \tag{4.16}$$

and their symplectic trace $C^{ij}p_{ij} = 0$ are invariant under $\mathrm{SL}(2,\mathbb{C})$.

Noting that $\mathrm{SL}(2,\mathbb{C})$ is the complexification of $\mathrm{SU}(2)$ [34], we can gauge fix such invariance precisely by setting the $\mathrm{SU}(2)$ $D$-term to vanish

$$D_{\mathrm{SU}(2)} = 0 \to \begin{cases} \sum_{i=1}^{4} |A^i|^2 = \sum_{i=1}^{4} |B^i|^2 \\ \sum_{i=1}^{4} A^i B^{*i} = 0\,. \end{cases} \tag{4.17}$$

The above discussion implies that the correct gauge group $\mathcal{G}$ to be used for $N$ coincident branes should reduce to $\mathrm{SU}(2)$ for $N = 1$. Hence, choosing $\mathcal{G}$ to reduce for $N = 1$ to $\mathrm{SU}(2)$, equation (4.17) fixes the $\mathrm{SL}(2,\mathbb{C})$ residual invariance. This gauge fixing is quite analogous to the one used in [5], where the complexification of the $\mathrm{U}(1)$ residual symmetry in the solution of the cone on $T^{11}$ is given by a complex rescaling of the relevant variables.

Given the above information, we can try to guess $\mathcal{G}$, when $N > 1$. The product of two unitary group is excluded, since the coordinate, as in $T^{11}$ (but not on the spheres), appears in the KK spectrum and it is a gauge singlet so that the spinor $S^\alpha_i$ must be in a pseudoreal representation of $\mathcal{G}$ (In a pseudoreal representation $C\bar{S} = S$, $C^2 = -1$ and thus the gauge singlet is contained in the antisymmetric product $(S \times S)_{\mathrm{asy}}$).

We are thus led as a minimal choice to the product of the two non-simply laced groups

$$\mathcal{G} = \mathrm{USp}(2N) \times \mathrm{O}(2N-1)\,. \tag{4.18}$$

[6] Curiously (4.15) is analogous to the moduli space of an $\mathrm{SU}(2)$ $\mathcal{N} = 2$ gauge theory with hypermultiplets with two flavours [33].

The rationale for this choice is that, if we take the singleton $S_i$ to be in the bifundamental representation of $\mathcal{G}$, then, since $S^i$ is in the $\mathbf{4}$ of Sp(4), the coordinate

$$z^a = \mathrm{Tr}(S^t \Gamma^a S) \tag{4.19}$$

is non-zero only if the gauge group contains a factor USp($2N$); the other factor must then be the orthogonal group O($2N-1$). Indeed, the orthogonal group, having a symmetric invariant metric, assures a non-zero value for $z^a$ and moreover its order is fixed to be $2N-1$ because of the condition $\mathcal{G}_{\langle \mathrm{vac} \rangle} = \mathrm{SU}(2)$.

Such groups usually arise when one deals with orientifold projections [30] and for the present case $N$ refers correctly to the total number of branes before mirroring.

From the chain decomposition

$$\begin{aligned}
\mathrm{USp}(2N) \times \mathrm{O}(2N-1) &\to \mathrm{USp}(2) \times \mathrm{USp}(2(N-1)) \times \mathrm{O}(2(N-1)) \\
&\to \mathrm{USp}(2) \times \mathrm{U}(N-1) \times \mathrm{U}(N-1) \\
&\to \mathrm{USp}(2) \times \mathrm{U}(N-1)_{diag} \to \mathrm{USp}(2) \times \mathrm{U}(1)^{N-1}
\end{aligned} \tag{4.20}$$

we can retrieve the phase where all but one brane are free to move at smooth points over the cone. Looking at the chain (4.20), we see that, by the first decomposition, we get

$$S^i = \left( \begin{array}{cc|c} A^i & B^i & 0 \;\dots\; 0 \\ \hline 0 & 0 & \\ \vdots & \vdots & S^i_{A\Lambda} \\ 0 & 0 & \end{array} \right), \tag{4.21}$$

where the upper left block is a $1 \times 2$ matrix, the lower right block has indices $A, \Lambda = 1, \dots, 2N-2$, while the off-diagonal blocks are rectangular $1 \times (2N-2)$ and $2 \times (2N-2)$ zero matrices.

Since USp($2(N-1)$) and O($2(N-1)$) both contain a U($N-1$) subgroup under which they both decompose as $(N-1) \oplus (\overline{N-1})$, we have $A \to a, \bar{a}$ and $\Lambda \to \alpha, \bar{\alpha}$ ($a, \bar{a}, \alpha, \bar{\alpha} = 1, \dots, N-1$). Correspondingly, the lower right $(2N-2) \times (2N-2)$ subblock of $S^i$ becomes

$$S^i_{A\Lambda} = \left( \begin{array}{c|c} S^i_{a\alpha} & S^i_{a\bar{\alpha}} \\ \hline S^i_{\bar{a}\alpha} & S^i_{\bar{a}\bar{\alpha}} \end{array} \right) \tag{4.22}$$

which derives from the second step of the chain (4.20). Further going to the diagonal U($N-1$), we have $S^i_{a\alpha} = S^i_{\bar{a}\bar{\alpha}} = 0$ and setting $S^i_{a\bar{\alpha}} = A^i$, $S^i_{\bar{a}\alpha} = B^i$ we find

$$S^i = \begin{pmatrix} 0 & A^i \\ B^i & 0 \end{pmatrix}. \tag{4.23}$$

When we consider a generic vacuum configuration $\mathrm{U}_{diag}(N-1) \to \mathrm{U}(1)^{N-1}$, the $A^i$, $B^i$ subblocks reduce to diagonal (commuting) matrices in the Cartan subalgebra.

We remark that it is likely that there exist just *one singleton* $S^i$ and that $A^i$ and $B^i$ are just specific components of these $S^i$. Indeed, promoting $A^i$ and $B^i$ to two independent singletons $S^i, T^i$, would imply that equation (4.19) admits the baryonic symmetry

$$\begin{aligned} S^i &\to S^i e^{i\alpha} \\ T^i &\to T^i e^{-i\alpha} \,. \end{aligned} \tag{4.24}$$

The baryonic symmetry is related to the existence of U(1) Betti multiplets in the KK spectrum [31, 10, 23], which only occur if there are non-trivial Betti numbers $b_i$, $i \neq 1, 7$. However, $V_{(5,2)}$ has the same real homology the seven-sphere $S^7$, and thus a continuous baryonic symmetry is ruled out.

Thus we propose that the CFT describing a large number of $M2$-branes on the (4.1) singularity is given by the infrared fixed point of an $\mathrm{USp}(2N) \times \mathrm{O}(2N-1)$ gauge theory where the basic degrees of freedom are chiral multiplets $S^i$ lying in the **4** of SO(5), with $R$-symmetry charge $Q = 1/2$ (or $y = 1/3$) and in the $(2N\,,\, 2N-1)$ irrep. of the gauge group. In the brane construction of gauge theories usually there can be other matter fields in symmetric and antisymmetric representations. We assume here that such representations decouple at the conformal IR fixed point.

The chiral fields (singletons) of the conformal field theory have $\Delta = |y| = 1/3$. This means that flowing to the interacting point they acquire an anomalous dimension $\gamma = -1/6$. This makes the conformal dimension violate the unitarity bound $\Delta \geq 1/2$, but since the singleton field is not a gauge group singlet it is not an observable of the theory. The analogous phenomenon occurs for the five-dimensional case $T^{11}$, where $\Delta_{A,B} = 3/4 < 1$ and for the proposed CFT's dual to the seven-dimensional manifolds $M^{111}$ and $Q^{111}$ [23].

As already remarked, the gauge theory exists only in the ultraviolet limit where it is not conformal and where the gauge vector potential, which is a singlet of the matter group SO(5) is in the adjoint representation of $\mathrm{USp}(2N) \times \mathrm{O}(2N-1)$. We could dualize it, at least in the Coulomb branch, and then reintroduce it in the CFT. However, from the KK analysis, we see that we have no states corresponding to products of this true singleton field (with $\Delta = 1/2$) and therefore we have no coordinates for the Coulomb branch.

As we will see later, it is also essential to introduce a superpotential whose jacobian ideal gives the needed vanishing relations for the correct matching of the chiral primaries with the supergravity hypermultiplets. This is given generically by the sixth power of the singleton fields

$$\mathcal{W}(S_i) = C_{ijklmn}\, \mathrm{Tr}(S^i S^j S^k S^l S^m S^n) \,. \tag{4.25}$$

where the tensor $C_{ijklmn}$ is constructed by an appropriate linear combination of products of three Sp(4) invariant metrics $C_{ij}$. It should probably be made of a combination of the following structures

$$\mathrm{Tr}\left[(SS)\,(SS)\,(SS)\right] \tag{4.26}$$

$$\mathrm{Tr}\left[(SS)\,S\Gamma^a S\,S\Gamma_a S\right] \tag{4.27}$$

$$\mathrm{Tr}\left[S\Gamma^a S\,S\Gamma^{bc} S\,S\Gamma^{de} S\right]\epsilon_{abcdef} \tag{4.28}$$

$$\mathrm{Tr}\left[(SS)\,S\Gamma^{ab} S\,S\Gamma_{ab} S\right] \tag{4.29}$$

$$\mathrm{Tr}\left[S\Gamma^{ab} S\,S\Gamma_{bc} S\,S\Gamma^{c}_{\ a} S\right] \tag{4.30}$$

$$\mathrm{Tr}\left[S\Gamma^{a} S\,S\Gamma^{b} S\,S\Gamma_{ab} S\right]. \tag{4.31}$$

Let us consider the previous structures for $N = 1$, when we can drop the trace symbol. We easily see that they are the six a priori existing singlets which can be obtained from the product of six **4** spinor representations of Sp(4). Next we note that when (4.23) holds, all the above structures are given by products of three $A^i$ and three $B^i$ contracted with three $C_{ij}$ tensors. Actually, there is just one possible Sp(4) invariant that can be built, namely

$$(A^i B^j C_{ij})^3\,. \tag{4.32}$$

Furthermore, the first four structures (4.26)–(4.29) are antisymmetric under the exchange of $A^i$ and $B^i$ while the last two are symmetric. That means that, by Fierz identities, (4.26)–(4.29) must be proportional to each other, while the other two must vanish identically.

## 5. AdS/CFT correspondence

OSp(4|2) **conformal superfields.** A generic OSp(4|2) representation [24] is labelled by three quantum numbers, according to the OSp(4|2) $\sim$ SO(2) $\times$ SO(3) $\times$ U$_R$(1) decomposition of the supergroup. They are the energy $\Delta$, the spin $s$ and the $R$-charge $y$.

This generic representation is unitary if

$$\Delta \geq 1 + s + |y|\,, \tag{5.1}$$

while short chiral representations can occur for

$$\Delta = |y| \geq \frac{1}{2}\,. \tag{5.2}$$

Like in the SU(2, 2|1) case [9], at the threshold of the unitarity bound (5.1), we can obtain short representations. These BPS-saturated states correspond to short superfields which thus satisfy some differential constraint.

Operators with protected dimensions are related to such shortenings and they fall in three categories:

- **Chiral superfields**: They occur when $\Delta = |y|$ and satisfy the condition

$$\overline{D}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0\,, \tag{5.3}$$

or $D_\alpha\Phi(x,\theta,\bar\theta) = 0$ for anti-chiral ones.

- **Conserved currents**: They occur when $\Delta = 1 + s$ and satisfy

$$D^{\alpha_1}J_{\alpha_1\dots\alpha_{2s}}(x,\theta,\bar\theta) = \overline{D}^{\alpha_1}J_{\alpha_1\dots\alpha_{2s}}(x,\theta,\bar\theta) = 0\,, \qquad \text{if } s\neq 0 \tag{5.4}$$

or

$$D^2J(x,\theta,\bar\theta) = \overline{D}^2J(x,\theta,\bar\theta) = 0\,, \qquad \text{for } s = 0\,. \tag{5.5}$$

- **Semiconserved currents**: They occur when $\Delta = 1 + s + |y|$ and satisfy

$$\begin{aligned}\overline{D}^{\alpha_1}L_{\alpha_1\dots\alpha_{2s}}(x,\theta,\bar\theta) &= 0\,, \qquad s\neq 0\\ \overline{D}^2L(x,\theta,\bar\theta) &= 0\,, \qquad s = 0\,,\end{aligned} \tag{5.6}$$

if left-semiconserved, or the conjugate conditions if right-semiconserved.

It is trivial to see that a right and left semi-conserved superfield is also conserved.

**The protected operators.** From the CFT point of view, we expect to have chiral operators corresponding to the wave-functions of the conifold [5]. Such operators are given by

$$\mathrm{Tr}\,\phi^k \equiv \mathrm{Tr}\,(z^{a_1},\dots,z^{a_k})\,C_{a_1\dots a_k} \tag{5.7}$$

with $C_{a_1\dots a_k}$ a completely symmetric and traceless rank $k$ tensor. They have $\Delta = y = \frac{2}{3}k$.

Surely, there should be a conserved current related to the global SO(5) symmetry, which should be a singlet of the gauge and $R$-symmetry group and that we can identify as

$$J^{ab} \equiv \bar S\Gamma^{ab}S\,. \tag{5.8}$$

This $J^{ab}$ should be massless and satisfy $\bar D^2J^{ab} = D^2J^{ab} = 0$. Its conformal dimension is therefore $\Delta = 1$.

Another operator with protected dimension we certainly expect is given by the stress-energy tensor

$$J_{\alpha\beta} = \bar D_\alpha\bar S D_\beta S + \bar D_\beta\bar S D_\alpha S + i\bar S\,\overset{\leftrightarrow}{\not\partial}_{\alpha\beta}S\,, \tag{5.9}$$

which has $\Delta = 2$, $y = 0$ and satisfies $D^\alpha J_{\alpha\beta} = \bar D^\alpha J_{\alpha\beta} = 0$.

It is now trivial to see that we should also expect KK supergravity states corresponding to the following semi-conserved superfields

$$\mathrm{Tr}(J_{\alpha\beta}\phi^k) \qquad \text{and} \qquad \mathrm{Tr}(J^{ab}\phi^k) \tag{5.10}$$

(or the conjugate ones).

It seems more problematic to find the appropriate singleton combinations which appear as semiconserved spin 1/2 superfields in the CFT corresponding to short gravitino multiplets on the supergravity side. In the theory at hand there is no field like the $W_\alpha$ of the $T^{11}$ case [5, 9] and thus there is no natural candidate for these operators. We also have to be careful not to use simple descendants of primary operators and this makes the task more difficult. Anyway, once we have the isometry group quantum numbers, from the KK analysis, we can see that the appropriate combinations of the $S_i^\alpha$ are uniquely fixed, and will be written explicitely below.

**The correspondence.** Given the structure of the OSp(4|2) multiplets of eleven-dimensional supergravity compactified on $AdS_4 \times V_{(5,2)}$, we can make the comparison between these results and the CFT predictions. We can also make use of these results to explicitly determine the expression of the fermionic operators related to the short gravitino multiplets.

Along the lines of the five-dimensional case of type-IIB supergravity on $AdS_5 \times T^{11}$ [9, 10], we look for rational conformal dimensions occurring in the KK multiplets and see whether they correspond to the right shortenings needed to be related to the previously described conformal operators.

From the energy values of the multiplets, it is easy to see that a rational conformal dimension can be obtained only if $H_0+36$ or $H_0+24$ are squares of rational numbers.

As in the $T^{11}$ case, we obtain rationality when we saturate the bound on the $R$-charge of a given harmonic, i.e. when in the Young tableaux all the boxes which can be charged have the same $R$-charge. This occurs for the representations[7] $[k\,,\,0]_{\frac{2}{3}k}$ of $\mathrm{SO}(5)_{\mathrm{U}_R(1)}$ in the $H_0 + 36$ case and for $[k\,,\,0]_{1+\frac{2}{3}k}$ in the $H_0 + 24$ case. The corresponding square roots are given by $6 + \frac{8}{3}k$ and $4 + \frac{8}{3}k$ respectively. We have solved the rationality constraint for the more generic case of $[m+n+k\,,\,m]_{1+\frac{2}{3}k}$, and we have found that there are two other infinite series of operators with rational dimension, for $m$ and $n$ satisfying the following relations

$$\begin{aligned} m^2 - n^2 - 2mn - 3n - m &= 0\,, \qquad \text{for } H_0 + 36\,, \\ m^2 - n^2 + 2m(1-n) &= 0\,, \qquad \text{for } H_0 + 24\,. \end{aligned} \tag{5.11}$$

This gives sequences of numbers with no simple rationale. Anyway we will see that as for $T^{11}$, beside the case $m = n = 0$, only another couple of $\mathrm{SO}(5) \times \mathrm{U}_R(1)$ quantum

[7]In the form $[\text{Young indices}]_{\text{charge}} = [M+N, M]_y$.

numbers, is related to shortenings, while all the others correspond to the rational long multiplets partially noticed in [7] and completely clarified in [9]. Here these couples are $m=1, n=0$ and $m=1, n=1$ respectively.

Let us now introduce these conditions on the $\mathrm{SO}(5)\times \mathrm{U}_R(1)$ quantum numbers in the $\Delta$ values of the supergravity multiplets and see when the shortening occurs.

We start with the graviton and vector multiplets for which we have some expectations to be verified and then pass to the gravitino multiplets. The graviton multiplet has

$$\Delta = \frac{1}{2}+\frac{1}{4}\sqrt{H_0+36}\,. \tag{5.12}$$

If the $\mathrm{SO}(5)\times \mathrm{U}_R(1)$ irrep is $[k\,,\,0]_{\frac{2}{3}k}$, it reduces to

$$\Delta = 2+\frac{2}{3}k\,, \tag{5.13}$$

which is the shortening condition $\Delta = 1+s+|y|$ related to the protected operator (5.10) corresponding to the massless and short graviton multiplets.

It can be easily seen that also in the $[k+1\,,\,1]_{\frac{2}{3}k}$ case, it is obtained a rational state with $\Delta = 3+\frac{2}{3}k$. These states do not satisfy the shortening condition $\Delta = 1+s+|y|$, but they can be put in correspondence with the rational non supersymmetry protected operators[8]

$$\mathrm{Tr}(J_{\alpha\beta}J^{ab}\phi^k)\,. \tag{5.14}$$

For the vector II,

$$\Delta = -\frac{3}{2}+\frac{1}{4}\sqrt{H_0+36}\,. \tag{5.15}$$

If we choose the $[k\,,\,0]_{\pm\frac{2}{3}k}$ irrep., we obtain states with

$$\Delta = \frac{2}{3}k\,, \tag{5.16}$$

which are hypermultiplet ($\Delta = |y|$) states associated to the $\phi^k$ operators. When the $G$-irrep is $[k+1\,,\,1]_{\frac{2}{3}k}$, we obtain again a shortening of the multiplet. Its anomalous dimension is given by

$$\Delta = 1+\frac{2}{3}k\,, \tag{5.17}$$

and is related to the massless gauge vector multiplet of the SO(5) matter group or to short vector multiplets corresponding to $\mathrm{Tr}(J^{ab}\phi^k)$ operators.

The other type of vector multiplets we have found never undergo shortening, but we can easily find the CFT rational long operators. Their anomalous dimension is

$$\Delta = \frac{5}{2}+\frac{1}{4}\sqrt{H_0+36}\,, \tag{5.18}$$

[8] Here and in the following the conformal operators have to be understood as projected along the SO(5) Young tableaux of the corresponding KK state.

which for the $[k\,,\,0]_{\pm\frac{2}{3}k}$ irreps reduces to

$$\Delta = 4 + \frac{2}{3}k \tag{5.19}$$

and for the $[k+1\,,\,1]_{\pm\frac{2}{3}k}$ case reduces to

$$\Delta = 5 + \frac{2}{3}k\,. \tag{5.20}$$

It is easy to see that the related CFT operators are given by

$$\mathrm{Tr}(J_{\alpha\beta}J^{\alpha\beta}\phi^k) \tag{5.21}$$

and

$$\mathrm{Tr}(J_{\alpha\beta}J^{\alpha\beta}J^{ab}\phi^k)\,. \tag{5.22}$$

Let us now examine the shortening conditions for the gravitino multiplets. Type-I gravitino has

$$\Delta = -\frac{1}{2} + \frac{1}{4}\sqrt{H_0+24}\,, \tag{5.23}$$

which, for the $[k\,,\,0]_{1+\frac{2}{3}k}$ irreps reduces to

$$\Delta = \frac{1}{2} + \frac{2}{3}k = -\frac{1}{2} + |y|\,. \tag{5.24}$$

This does not correspond to a shortening condition, but nevertheless satisfies the unitarity bound $\Delta \geq 1 + s + |y|$.

We can obtain unitary multiplets when the $G$ quantum numbers are $[k{+}2\,,1]_{1+\frac{2}{3}k}$. In this case indeed

$$\Delta = \frac{5}{2} + \frac{2}{3}k = 1 + s + |y| \tag{5.25}$$

and therefore we obtain short gravitino multiplets.

For type-II gravitino we have

$$\Delta = \frac{3}{2} + \frac{1}{4}\sqrt{H_0+24}\,, \tag{5.26}$$

which, for the $[k\,,\,0]_{1+\frac{2}{3}k}$ irreps reduces to

$$\Delta = \frac{5}{2} + \frac{2}{3}k \tag{5.27}$$

undergoing shortening, and for $[k+2\,,\,1]_{1+\frac{2}{3}k}$

$$\Delta = \frac{9}{2} + \frac{2}{3}k \tag{5.28}$$

gives long rational multiplets.

Having the OSp(4|2) and matter group quantum numbers, we can try to guess the corresponding conformal operators. For $k=0$, those related to the short type-I gravitinos are given by

$$\mathrm{Tr}\, L_\alpha = \mathrm{Tr}\left[\left(\bar{S}\Gamma^a S\, \bar{D}_\alpha \bar{S}\Gamma^{bc} S - \bar{D}_\alpha \bar{S}\Gamma^a S\, \bar{S}\Gamma^{bc} S\right)\right], \tag{5.29}$$

while those related to short type-II gravitinos are

$$\mathrm{Tr}\, X_\alpha = \mathrm{Tr}\left[S\Gamma_a S\ \left(\bar{S}\Gamma_b \bar{S}\, \bar{D}_\alpha \bar{S}\Gamma^{ab} S - 2\bar{S}\Gamma_b \bar{D}_\alpha \bar{S}\, \bar{S}\Gamma^{ab} S\right)\right], \tag{5.30}$$

which become $\mathrm{Tr}(L_\alpha \phi^k)$ and $\mathrm{Tr}(X_\alpha \phi^k)$ for the generic cases (5.25) and (5.27) respectively. Equations (5.29) and (5.30) are easily seen to obey the semi-conservation condition (5.6).

We point out that in the $L_\alpha$ operator, only the irreducible $a[bc]$ representation survives once we use the $\Gamma$-matrices identities and the $D$-term equations.

Let us note explicitly that, as anticipated in the introduction the type-II short gravitino multiplet $\mathrm{Tr}\, X_\alpha$ has a lowest component of $R$-symmetry $y=1$, so that its $\bar{\theta}$ component, which we call $\tilde{W}_\alpha$, has $y=0$. Moreover, it is a singlet under SO(5) so that in the infrared limit $\tilde{W}_\alpha$ has the same quantum numbers (apart from the conformal dimension) as the original gauge field in the ultraviolet limit.

We may compare this result with the gravitino sector superfield of the four dimensional SCFT dual to the $T^{11}$ compactification of type-IIB theory and called $L^{1k}_{\dot{\alpha}}$ in [9]. There, the vector field strength superfield $W_\alpha$ is the singleton of the conformal theory, so that it does not appear in the spectrum of $T^{11}$ compactification. In the present case instead, $\mathrm{Tr}[X_\alpha \phi^k]$ does indeed appear in the spectrum of $V_{(5,2)}$ being a composite field of singletons. Furthermore, $L^{1k}_{\dot{\alpha}}$ does not exist for $k=0$, since in this case it would reduce to [9]

$$L^{10}_{\dot{\alpha}} = \mathrm{Tr}(e^V W_{\dot{\alpha}} e^{-V}), \tag{5.31}$$

which vanishes identically while $\mathrm{Tr}[X_\alpha \phi^k]$ is different from zero even for $k=0$.

Finally, for type-II gravitinos, we have in addition states with $\Delta = \frac{9}{2} + \frac{2}{3}k$ corresponding to long rational multiplets, which can be written as

$$\mathrm{Tr}\left[(S\bar{D}_\beta \bar{S})(S\bar{D}^\beta \bar{S})\bar{L}_\alpha \phi^k\right]. \tag{5.32}$$

## 6. Summary

In order to collect our results, we present a table where we list the multiplet type as well as spin, representation and energy of the highest states for $M$-theory compactified on the Stiefel manifold, and match them with the boundary conformal superfields. These results merely rely on the AdS/CFT correspondence.

It remains an open problem to make an explicit construction of the ultraviolet description of the underlying field theory in terms of $D2$-brane gauge theory.

| s | $\Delta$ | $\mathrm{SO}(5)_{\mathrm{U}_R(1)}$ irreps | multiplet | Conformal superfield |
|---|---|---|---|---|
| 1 | $2+\frac{2}{3}k$ | $[k\,,\,0]_{\frac{2}{3}k}$ | short graviton[9] | $T_{\alpha\beta}\phi^k$ |
| 1 | $3+\frac{2}{3}k$ | $[k+1\,,\,1]_{\frac{2}{3}k}$ | long graviton | $J^{ab}\,T_{\alpha\beta}\phi^k$ |
| 1/2 | $\frac{1}{2}+\frac{2}{3}k$ | $[k\,,\,0]_{1+\frac{2}{3}k}$ | non unitary | |
| 1/2 | $\frac{5}{2}+\frac{2}{3}k$ | $[k\,,\,0]_{1+\frac{2}{3}k}$ | short gravitino II | $X_\alpha\phi^k$ |
| 1/2 | $\frac{5}{2}+\frac{2}{3}k$ | $[k+2\,,\,1]_{1+\frac{2}{3}k}$ | short gravitino I | $L_\alpha\phi^k$ |
| 1/2 | $\frac{9}{2}+\frac{2}{3}k$ | $[k+2\,,\,1]_{1+\frac{2}{3}k}$ | long gravitino I | $(S\bar{D}_\beta\bar{S})(S\bar{D}^\beta\bar{S})\bar{L}_\alpha\phi^k$ |
| 0 | $\frac{2}{3}k$ | $[k\,,\,0]_{\frac{2}{3}k}$ | hypermultiplet | $\phi^k$ |
| 0 | $1+\frac{2}{3}k$ | $[k+1\,,\,1]_{\frac{2}{3}k}$ | short vector[9] II | $J^{ab}\,\phi^k$ |
| 0 | $4+\frac{2}{3}k$ | $[k\,,\,0]_{\frac{2}{3}k}$ | long vector I | $T_{\alpha\beta}T^{\alpha\beta}\,\phi^k$ |
| 0 | $5+\frac{2}{3}k$ | $[k+1\,,\,1]_{\frac{2}{3}k}$ | long vector I | $T_{\alpha\beta}T^{\alpha\beta}\,J^{ab}\,\phi^k$ |

**Table 7:** Summary of the correspondence between KK states and conformal fields.

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## A. Rescaled connection and curvature on G/H

In this appendix we present an algebraic technique to derive the rescaled connection and curvature on a coset manifold given the structure constants of the $G$, $H$ and $G/H$ groups that generalizes the formulae of [35] and that has recently appeared in [37]. The $a, b$ are the coset indices, $i, j$ are the $H$ indices while $e^a$ and $\omega^i$ are the vielbeins.

[9]Massless for $k = 0$.

The Maurer-Cartan equations for $e^a$ and $\omega^i$ are

$$\begin{aligned} de^a + \frac{1}{2} C^a{}_{bc} e^b e^c + C^a{}_{bi} e^b \omega^i &= 0\,, \\ d\omega^i + \frac{1}{2} C^i{}_{bc} e^b e^c + \frac{1}{2} C^i{}_{jk} \omega^j \omega^k &= 0\,. \end{aligned} \tag{A.1}$$

Under a rescaling of $e^a$, equations (A.1) become:

$$\begin{aligned} de^a + \frac{1}{2} \frac{r(b) r(c)}{r(a)} C^a{}_{bc} e^b e^c + \frac{r(b)}{r(a)} C^a{}_{bi} e^b \omega^i &= 0\,, \\ d\omega^i + \frac{1}{2} r(a) r(b)\, C^i{}_{bc} e^b e^c + \frac{1}{2} C^i{}_{jk} \omega^j \omega^k &= 0\,. \end{aligned} \tag{A.2}$$

The connection one-form on $G/H$ can be defined by

$$de^a - \mathcal{B}^a{}_b e^b = 0\,. \tag{A.3}$$

Combining (A.3) and (A.2) yields

$$\mathcal{B}^a{}_b = \frac{1}{2} \frac{bc}{a} C^a{}_{bc} e^c + \frac{r(b)}{r(a)}\, C^a{}_{bi} \omega^i + K^a{}_{bc} e^c\,, \tag{A.4}$$

where $K^a{}_{bc}$, symmetric in $b, c$, is determined by the requirement of antisymmetry of $\mathcal{B}$.

Thus the antisymmetric connection $\mathcal{B}$ is given by

$$\mathcal{B}^a{}_b = \frac{1}{2} \mathbb{C}^a{}_{bc} e^c \frac{r(b)}{r(a)}\, C^a{}_{bi} \omega^i\,, \tag{A.5}$$

where

$$\mathbb{C}^a{}_{bc} \equiv \frac{r(b) r(c)}{r(a)} C^a{}_{bc} + \frac{r(a) r(c)}{r(b)} C^f{}_{ce} \eta^{ea} \eta_{fb} - \frac{r(a) r(b)}{r(c)} C^g{}_{fb} \eta^{af} \eta_{cg}\,. \tag{A.6}$$

The Riemann curvature is defined in terms of the connection as

$$R^a{}_b \equiv d\mathcal{B}^a{}_b - \mathcal{B}^a{}_c \mathcal{B}^c{}_b \equiv R^a{}_{bde} e^d e^e\,. \tag{A.7}$$

Substituting the definition of $\mathcal{B}$ in terms of the structure constants given above, and using the Maurer-Cartan equations for the differentiated vielbeins and Jacobi identities for products of structure constants, one arrives at

$$R^a{}_{bde} = -\frac{1}{4} \mathbb{C}^a{}_{bc} C^c{}_{de} \frac{r(d) r(e)}{r(c)} - \frac{1}{2} C^a{}_{bi} C^i{}_{de} r(d) r(e) - -\frac{1}{8} \mathbb{C}^a{}_{cd} \mathbb{C}^c{}_{be} + \frac{1}{8} \mathbb{C}^a{}_{ce} \mathbb{C}^c{}_{bd}\,. \tag{A.8}$$

This form of the Riemann tensor is more general than the one presented in [35], where the final result depended only on the $C^a{}_{bc}$ and not on the $\mathbb{C}^a{}_{bc}$ due to the hypothesis that the Killing metric be completely diagonal. In our case instead the mixed components $\gamma_{ia}$ are non-zero, while the condition that within an isotropy-irreducible subspace the Killing metric is proportional to $\delta_{ab}$ still holds. This is necessary to ensure the antisymmetry of the connection $\mathcal{B}^{ab}$.

It is straightforward to verify that when the Killing metric is diagonal, the $\mathbb{C}^a{}_{bc}$ reduces to the combination $\left(\begin{smallmatrix} a\ b \\ c \end{smallmatrix}\right) C^a{}_{bc}$ of [35].

## B. The reduction of SO(7) under SO(3) × SO(2).

In this section we reduce the SO(7) indices to $H$-irreducible indices.

The embedding of SO(3) × SO(2) in SO(7) is defined by

$$(T_H)^{ab} = (C_H{}^{\alpha})^{\beta} (T_{\alpha\beta})^{ab}\,, \tag{B.1}$$

relating the generators of $H$ in an SO(7) irrep to the generators of SO(7) in the same irrep through the structure constants. In the vector representation of SO(7) one has

$$(T_{\alpha\beta})^{\gamma\delta} = -\delta^{\gamma\delta}_{\alpha\beta}$$

and therefore

$$(T_H)^{\alpha\beta} = (C_H{}^{\alpha})^{\beta}\,.$$

Using the expressions for the structure constants one obtains

$$(N)_{\alpha}{}^{\beta} = \left(\begin{array}{c|c|c} 0 & \delta_m{}^{\hat m} & 0 \\ \hline -\delta_{\hat m}{}^{m} & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{B.2}$$

and

$$(J^i)^{\alpha\beta} = \left(\begin{array}{c|c|c} \epsilon^{imn} & 0 & 0 \\ \hline 0 & \epsilon^{i\hat m\hat n} & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{B.3}$$

Thus the SO(7) vector reduces under $H$ as

$$\mathbf{7} \to \mathbf{3}_1 \oplus \mathbf{3}_{-1} \oplus \mathbf{1}_0\,, \tag{B.4}$$

where the first number labels the SO(3) irrep, while the second one is the $\mathrm{U}(1)_H$ charge.

To construct them in the spinor representations we use the following $\gamma$ matrices:

$$\begin{aligned} \gamma^m &= \left\{ i\sigma^1 \otimes \sigma^1 \otimes \sigma^2, i\sigma^1 \otimes \sigma^2 \otimes \mathbb{1}, -i\sigma^1 \otimes \sigma^3 \otimes \sigma^2 \right\} \\ \gamma^{\hat m} &= \left\{ -i\sigma^2 \otimes \sigma^2 \otimes \sigma^1, i\sigma^2 \otimes \sigma^2 \otimes \sigma^3, -i\sigma^2 \otimes \mathbb{1} \otimes \sigma^2 \right\} \\ \gamma^7 &= i\sigma^3 \otimes \mathbb{1} \otimes \mathbb{1}\,. \end{aligned} \tag{B.5}$$

The charge conjugation matrix is

$$C = \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1}\,. \tag{B.6}$$

The $N$ generator in the spinor rep. is thus

$$N = \frac{1}{2}\gamma^m \gamma^{\hat{m}} \delta_{m\hat{m}} = -\frac{i}{2} \left( \begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -3 & & & & \\ \hline & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & 3 \end{array} \right), \tag{B.7}$$

and the $J^i$ are

$$J^i \sim \left( \begin{array}{c|c} \begin{array}{c|c} \epsilon^{ijk} & 0 \\ \hline 0 & 0 \end{array} & \mathbb{O}_4 \\ \hline \mathbb{O}_4 & \begin{array}{c|c} \epsilon^{ijk} & 0 \\ \hline 0 & 0 \end{array} \end{array} \right), \tag{B.8}$$

so the eight-dimensional spinor representation of SO(7) reduces under the $H$ subgroup as

$$\mathbf{8} \to \mathbf{3}_{1/2} \oplus \mathbf{3}_{-1/2} \oplus \mathbf{1}_{3/2}\,. \tag{B.9}$$

We will decompose the eight-component Majorana spinor as $\binom{H}{\Lambda}$, where

$$\Lambda = \begin{pmatrix} \phi_{(3,1/2)} \\ \omega_{(1,-3/2)} \end{pmatrix}.$$

Since ours are Majorana spinors $C\Xi^* = \Xi$, $H^* = \Lambda$ and then our generic spinor is

$$\Xi = \begin{pmatrix} \phi^k_- \\ \omega_+ \\ \phi^k_+ \\ \omega_- \end{pmatrix}. \tag{B.10}$$

# References


[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[5] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[6] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, *Nucl. Phys.* **B 556** (1999) 89 [hep-th/9905104].

[7] S.S. Gubser, *Einstein manifolds and conformal field theories*, *Phys. Rev.* **D 59** (1999) 025006 [hep-th/9807164].

[8] D.P. Jatkar and S. Randjbar-Daemi, *Type IIB string theory on $AdS_5 \times T^{nn'}$*, *Phys. Lett.* **B 460** (1999) 281 [hep-th/9904187].

[9] A. Ceresole, G. Dall'Agata, R. D'Auria and S. Ferrara, *Spectrum of type-IIB supergravity on $AdS_5 \times T^{11}$: predictions on $N = 1$ SCFT's*, *Phys. Rev.* **D 61** (2000) 066001 [hep-th/9905226].

[10] A. Ceresole, G. Dall'Agata and R. D'Auria, *KK spectroscopy of type-IIB supergravity on $AdS_5 \times T^{11}$*, *J. High Energy Phys.* **11** (1999) 009 [hep-th/9907216].

[11] O. Aharony, Y. Oz and Z. Yin, *M-theory on $AdS_p \times S^{11-p}$ and superconformal field theories*, *Phys. Lett.* **B 430** (1998) 87 [hep-th/9803051].

[12] E. Halyo, *Supergravity on $AdS_{4/7} \times S^{7/4}$ and M-branes*, *J. High Energy Phys.* **04** (1998) 011 [hep-th/9803077].

[13] S. Minwalla, *Particles on $AdS_{(4/7)}$ and primary operators on $M(2/5)$ brane worldvolumes*, *J. High Energy Phys.* **10** (1998) 002 [hep-th/9803053].

[14] S. Ferrara, A. Kehagias, H. Partouche and A. Zaffaroni, *Membranes and fivebranes with lower supersymmetry and their AdS supergravity duals*, *Phys. Lett.* **B 431** (1998) 42 [hep-th/9803109];
R. Entin and J. Gomis, *Spectrum of chiral operators in strongly coupled gauge theories*, *Phys. Rev.* **D 58** (1998) 105008 [hep-th/9804060];
S. Sethi, *A relation between $N = 8$ gauge theories in three dimensions*, *J. High Energy Phys.* **11** (1998) 003 [hep-th/9809162];
M. Berkooz and A. Kapustin, *New IR dualities in supersymmetric gauge theory in three dimensions*, *J. High Energy Phys.* **02** (1999) 009 [hep-th/9810257].

[15] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[16] L. Castellani et al., *G/H M-branes and $AdS_{p+2}$ geometries*, *Nucl. Phys.* **B 527** (1998) 142 [hep-th/9803039].

[17] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull and B. Spence, *Branes at conical singularities and holography*, *Adv. Theor. Math. Phys.* **2** (1999) 1249 [hep-th/9808014].

[18] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[19] L. Castellani, L.J. Romans and N.P. Warner, *A classification of compactifying solutions for $D = 11$ supergravity*, *Nucl. Phys.* **B 241** (1984) 429.

[20] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, *Phys. Rep.* **130** (1986) 1.

[21] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective*, World Scientific 1991.

[22] O. Pelc and R. Siebelink, *The D2-D6 system and a fibered AdS geometry*, *Nucl. Phys.* **B 558** (1999) 127 [hep-th/9902045].

[23] D. Fabbri, P. Fré, C. Reina, A. Tomasiello, A. Zaffaroni and A. Zampa, *3d superconformal theories from Sasakian seven-manifolds: new nontrivial evidences for $AdS_4/CFT_3$*, hep-th/9907219.

[24] A. Ceresole, P. Fré and H. Nicolai, *Multiplet structure and spectra of $N = 2$ supersymmetric compactifications*, *Class. and Quant. Grav.* **2** (1985) 133.

[25] D. Fabbri, P. Fré, L. Gualtieri and P. Termonia, *M-theory on $AdS_4 \times M^{111}$: the complete* $\mathrm{OSp}(2|4) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$ *spectrum from harmonic analysis*, *Nucl. Phys.* **B 560** (1999) 617 [hep-th/9903036].

[26] C. Ahn, *$N = 2$ SCFT and M-theory on $AdS_4 \times Q^{1,1,1}$*, *Phys. Lett.* **B 466** (1999) 171 [hep-th/9908162].

[27] G. Dall'Agata, *$N = 2$ conformal field theories from M2-branes at conifold singularities*, *Phys. Lett.* **B 460** (1999) 79 [hep-th/9904198].

[28] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[29] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025 [hep-th/9808075].

[30] J. Park, R. Rabadan and A.M. Uranga, *Orientifolding the conifold*, hep-th/9907086; A.M. Uranga, *A new orientifold of $C^2/Z_n$ and six-dimensional RG fixed points*, hep-th/9910155.

[31] R. D'Auria and P. Fré, *Universal Fermi-Bose mass relations in KK supergravity and harmonic analysis on coset manifolds with Killing spinors*, *Ann. Phys. (NY)* **162** (1985) 372.

[32] C. Ahn and S.-J. Rey, *More CFTS and RG flows from deforming M2/M5-brane horizon*, hep-th/9911199.

[33] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[34] J. Bagger and J. Wess, *Supersymmetry and supergravity*, Princeton Series in Physics, 1992.

[35] L. Castellani, L.J. Romans and N.P. Warner, *Symmetries of coset spaces and Kaluza-Klein supergravity*, *Ann. Phys. (NY)* **157** (1984) 394.

[36] J. de Boer, K. Hori and Y. Oz, *Dynamics of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 500** (1997) 163 [hep-th/9703100].

[37] L. Castellani, *On $G/H$ geometry and its use in M-theory compactifications*, hep-th/9912277.

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